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Adapted linear approximation for singular integrals

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Abstract

Purpose: The purpose of this work is to present an approximation for singular integrals of Cauchy type kernel on a smooth-oriented contour.

Methods: For the method, we use a small modification of the spline functions in order to eliminate the singularity. Noting that this approximation is due to the idea of Sanikidze's *Approximate solution of singular integral equations in the case of closed contours of integrations*.

Results: This approximation represents a good approach for any singular integral given on the curve in the sense of Cauchy principal value.

Conclusions: This approximation is destined to solve numerically the singular integral equations with Cauchy type kernel on a smooth-oriented contour.

Keywords: Singular integral, Interpolation; Hőlder space, Hőlder condition, Approximation theory, Spline functions

2000 MSC: 45D05; 45E05; 45L05; 45L10; 65R20

Introduction

The method of numerical solution of singular integral equations of the first and second kind is convenient for application of electronic computers is necessary [1]. For equations of the first kind, many methods are developed in aerodynamics such as the method of discrete vortices. We find analogous equations in the theory of cracks.

The main consideration of the present work is the construction of a new approximation of a singular integral in order to use it for a numerical solution of singular integral equations.

$$a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

where, under Γ, we designate a smooth-oriented contour; t and t_0 are points on Γ; and $a(t), b(t), k(t,t_0)$ and $f(t)$ are given functions on Γ. Our schemes describe the quadrature method for the approximation of singular integral operator:

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt, \quad t, t_0 \in \Gamma, \tag{2}$$

with Cauchy kernel by a sequence of numerical integration operators. For the existence of the principal value of this integral for a given density $\varphi(t)$, we will need more than mere continuity. In other words, the density $\varphi(t)$ has to satisfy the Hölder condition $H(\mu)$ [2]. So, we note that singular integral operators of the first kind with Cauchy kernel have index zero. In particular, injective singular integral operators of the first kind are bijective and have bounded inverse.

Methods

Let $t = t(s) = x(s) + iy(s)$, where $s \in [a,b]$ is the parametric complex equation of the curve Γ with the respect to some parameter s. Consider that N is an arbitrary natural number; generally, we take it large enough and divide the interval $[a,b]$ into N equal subintervals of $[a,b]$:

$$[a,b] = \{a = s_0 < s_1 < \ldots\ldots < s_N = b\},$$

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be called I_1 to I_N, so that, we have $I_{\sigma+1} = [s_\sigma, s_{\sigma+1}]$.

$$s_\sigma = a + \sigma \frac{l}{N}\ ,\ l = b - a\ ,\ \sigma = 0, 1, 2, \ldots, N.$$

Further, fixing a natural number $M > 1$ and divide each of segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points,

$$s_{\sigma k} = s_\sigma + k\frac{h}{M},\ \ h = \frac{l}{N},\ \ k = 0, 1, \ldots, M.$$

In other words, we have, for the subinterval $[s_\sigma, s_{\sigma+1}]$, the following subdivision [1-7]:

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < \ldots\ldots < s_{\sigma M} = s_{\sigma+1}\}$$

Denoted by

$$t_\sigma = t(s_\sigma),\ t_{\sigma k} = t(s_{\sigma k});\quad \sigma = 0, 1, 2, \ldots, N;\quad k = 0, 1, \ldots, M.$$

It means that the interval $[a, b]$ is divided into $N_1 = MN$ equal subintervals.

For an arbitrary number $\sigma = 0, 1, 2, \ldots, N-1$, we define the spline function $S_1(\varphi; t, \sigma)$ dependents of φ, t and σ. This latter represents the linear approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. Noting that the interval $[t_\sigma, t_{\sigma+1}]$ is divided into subintervals $[t_{\sigma k}, t_{\sigma(k+1)}]$ of length $(t_{\sigma(k+1)} - t_{\sigma k})$ and interpolates the density function $\varphi(t)$ with respect to the values $\varphi(t_{\sigma k})$ and $\varphi(t_{\sigma(k+1)})$ at the points $t_{\sigma k}$ and $t_{\sigma(k+1)}$, respectively, with a linear polynomial, the following formula gives, for $t_{\sigma k} \le t \le t_{\sigma(k+1)}$:

$$\begin{aligned} S_1(\varphi; t, \sigma) =& \frac{(t_{\sigma(k+1)} - t)}{(t_{\sigma(k+1)} - t_{\sigma k})}\varphi(t_{\sigma k}) \\ &+ \frac{(t - t_\sigma)}{(t_{\sigma(k+1)} - t_{\sigma k})}\varphi(t_{\sigma(k+1)}). \end{aligned} \tag{3}$$

This spline function exists and is uniquely called a linear interpolating polynomial. Defined for an arbitrary numbers σ and ν such that $0 \le \sigma, \nu \le N-1$, the function $\beta_{\sigma\nu}(\varphi; t, t_0)$ depends on φ, t and t_0 by this equation:

$$\beta_{\sigma\nu}(\varphi; t, t_0) = \begin{cases} U(\varphi; t, \sigma) - V(\varphi; t_0, \sigma, \nu), t \ne t_0 \\ 0 \qquad\qquad\qquad\qquad\qquad t = t_0 \end{cases}, \tag{4}$$

where the points t and t_0 belong, respectively, to the arcs $\overset{\frown}{t_\sigma t_{\sigma+1}}$ and $\overset{\frown}{t_\nu t_{\nu+1}}$ with $\overset{\frown}{t_\alpha t_{\alpha+1}}$ designating the smallest arc with ends t_α and $t_{\alpha+1}$. The density φ represents still a given function on the curve Γ and of the class $H(\mu)$ [1,5,7]. The function $U(\varphi; t, \sigma)$ represents a modified linear interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ.

Indeed, for $t_{\sigma k} \le t \le t_{\sigma(k+1)}$, we put

$$\begin{aligned} U(\varphi; t, \sigma) =& \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}}\varphi(t_{\sigma k})\frac{t - t_0}{t_{\sigma k} - t_0} \\ &+ \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}}\varphi(t_{\sigma(k+1)})\frac{t - t_0}{t_{\sigma(k+1)} - t_0}, \end{aligned}$$

and the function $V(\varphi; t, \sigma, \nu)$ is given by

$$\begin{aligned} V(\varphi; t_0, \sigma, \nu) =& \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma(k+1)} - t)}{(t_{\sigma k} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \\ &+ \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})}, \end{aligned}$$

where the function φ represents a given function on the curve Γ of the class $H(\mu)$. Noting that $\psi_{\sigma\nu}(\varphi; t, t_0)$ is the quadratic approximation of the density $\varphi(t)$ at the point $t \in [t_\sigma, t_{\sigma+1}]$, $t_0 \in [t_\nu, t_{\nu+1}]$ and $0 \le \sigma, \nu \le N-1$ by

$$\psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0), \tag{5}$$

and replacing this latter in the singular integral (2)

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t - t_0}dt,$$

we obtain the following approximation:

$$\begin{aligned} S(\varphi, t_0) =& \frac{1}{\pi i}\int_\Gamma \frac{\psi_{\sigma\nu}(\varphi; t, t_0)}{t - t_0}dt \\ =& \varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\beta_{\sigma\nu}(\varphi; t, t_0)}{t - t_0}dt. \end{aligned} \tag{6}$$

Results and discussion

Theorem 1. *Let Γ be a smooth contour oriented, and let φ be a function density defined on Γ satisfies the Hölder condition $H(\mu)$, then the following estimation*

$$|\ F(t_0) - S(\varphi; t_0)\ | \le \max\left(\frac{C_1 \ln(MN)}{(MN)^\mu}, \frac{C_2}{N^\mu}\right)\ \ N,\ M > 1$$

holds, where the constant C_1, C_2 depends only of the contour Γ and the Holder's constant.

Proof. For any points $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_\nu, t_{\nu+1}]$ with the conditions $t_{\sigma k} \le t \le t_{\sigma(k+1)}$ and $t_{\nu k} \le t_0 \le t_{\nu(k+1)}$, we have the equation below:

$$\begin{aligned} \varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0) =& \varphi(t) - \varphi(t_0) \\ &- \{\frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}}\varphi(t_{\sigma k})\frac{t - t_0}{t_{\sigma k} - t_0} \\ &+ \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}}\varphi(t_{\sigma(k+1)})\frac{t - t_0}{t_{\sigma(k+1)} - t_0} \\ &- \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma(k+1)} - t)}{(t_{\sigma k} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \\ &- \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})}\}. \end{aligned} \tag{7}$$

□

Table 1 Exact principal value of singular integral, approximate calculation of the integral, and error for $N_1 = 30$ (Example 1)

$N_1 = 30$	Exact solution	Approximate solution	Error
	3.3415e-001 -2.3328e-002i	3.3402e-001 -2.2816e-002i	5.2769e-004
	3.3666e-001 -4.6999e-002i	3.3642e-001 -4.6499e-002i	5.5707e-004
	3.4106e-001 -7.1367e-002i	3.4068e-001 -7.0895e-002i	6.0630e-004
	3.4770e-001 -9.6807e-002i	3.4717e-001 -9.6386e-002i	6.8121e-004
	3.5714e-001 -1.2372e-001i	3.5642e-001 -1.2339e-001i	7.9148e-004

Taking into account expression (7), we get the following equation:

$$\frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt = \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}}\frac{\varphi(t)-\psi_{\sigma\nu}(\varphi;t,t_0)}{t-t_0}dt; \tag{8}$$

hence,

$$\begin{aligned}F(t_0)-S(\varphi;t_0) = \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma k}t_{\sigma(k+1)}} & \frac{\varphi(t)-\varphi(t_0)}{t-t_0} \\ & -\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0} \\ & +\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0} \\ & -\frac{S_1(\varphi;t_0,\nu)(t-t_0)(t_{\sigma(k+1)}-t)}{(t_{\sigma k}-t_0)(t_{\sigma(k+1)}-t_{\sigma k})} \\ & -\frac{S_1(\varphi;t_0,\nu)(t-t_0)(t-t_{\sigma k})}{(t_{\sigma(k+1)}-t_0)(t_{\sigma(k+1)}-t_{\sigma k})}\}\frac{1}{t-t_0}dt,\end{aligned}$$

seeing that the equalities $t_{\sigma k}-t_0=0$ and $t_{\sigma(k+1)}-t_0$ are possible only when $\sigma=\nu-1,\nu+1$ and ν. For the two first cases, the integral (8) exists when $t_{\sigma k}$ tends to t_0 or $t_{\sigma(k+1)}$ tends to t_0; on the other case, if $\sigma=\nu$, we can easily see that the function $\beta_{\sigma\sigma}(\varphi;t,t_0)$ contains $(t_{\sigma k}-t_0)$ and $(t_{\sigma(k+1)}-t_0)$ as factors for the points $t,\ t_0\in[t_\sigma,t_{\sigma+1}]$ with the conditions $t_{\sigma k}\le t,\ t_0\le t_{\sigma(k+1)}$, so we have the equation below:

$$\beta_{\sigma\sigma}(\varphi;t,t_0)=U(\varphi;t,\sigma)-V(\varphi;t_0,\sigma,\sigma).$$

Therefore,

$$\begin{aligned}\beta_{\sigma\sigma}(\varphi;t,t_0)= & \frac{(t_{\sigma(k+1)}-t)(t-t_0)}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma k}-t_0)}(\varphi(t_{\sigma k}) \\ & -S_1(\varphi;t_0,\sigma)) \\ & +\frac{(t-t_{\sigma k})(t-t_0)}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+1)}-t_0)}(\varphi(t_{\sigma(k+1)}) \\ & -S_1(\varphi;t_0,\sigma)).\end{aligned} \tag{9}$$

It is easy to see that

$$\beta_{\sigma\sigma}(\varphi;t,t_0)=(t-t_0)Q(\varphi;t,t_0),$$

where

$$\begin{aligned}Q(\varphi;t,t_0)= & \frac{(t_{\sigma(k+1)}-t)}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma k}-t_0)}(\varphi(t_{\sigma k}) \\ & -S_1(\varphi;t_0,\sigma)) \\ & +\frac{(t-t_{\sigma k})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+1)}-t_0)}(\varphi(t_{\sigma(k+1)}) \\ & -S_1(\varphi;t_0,\sigma)).\end{aligned}$$

Table 2 Exact principal value of singular integral, approximate calculation of integral, and error for $N_1 = 20$ (Example 2)

$N_1 = 20$	Exact solution	Approximate solution	Error
	-2.7101e-001 -1.4676e-001i	-2.6184e-001 -1.4383e-001i	9.6317e-003
	-1.6706e-001 -2.0230e-001i	-1.6597e-001 -1.9663e-001i	5.7670e-003
	-9.0558e-002 -2.0774e-001i	-9.2594e-002 -2.0463e-001i	3.7191e-003
	-3.9496e-002 -2.0259e-001i	-4.2143e-002 -2.0147e-001i	2.8746e-003
	3.9496e-002 -2.0259e-001i	3.6897e-002 -2.0468e-001i	3.3353e-003

Note: Many examples confirm the efficiency of this approximation.

Passing now to the estimation of the expression (8), for $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_\nu, t_{\nu+1}]$ with the conditions $\sigma \neq \nu-1, \nu+1$ and ν, we obtain the equation below:

$$\left| \begin{array}{l} \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma k}t_{\sigma(k+1)}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0} \\ -\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0} \\ +\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0} \\ -\frac{t-t_0}{t_{\sigma(k+1)}-t_{\sigma k}}(\frac{t_{\sigma(k+1)}-t}{t_{\sigma k}-t_0} \\ +\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_0})S_1(\varphi;t_0,\nu)\}\frac{1}{t-t_0}dt. \end{array} \right| = O(\frac{\ln MN}{M^\mu N^\mu}).$$

Indeed, it is clear that

$$\max_{t_0\in t_\nu \widehat{t_{\nu+1}}} \mid \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma k}}^{t_{\sigma(k+1)}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0} \mid = O(\frac{\ln MN}{M^\mu N^\mu});$$

also, we estimate the expression

$$\left| \begin{array}{l} \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma k}t_{\sigma(k+1)}} -\{\frac{t_{\sigma(k+1)}-t}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma k})\frac{t-t_0}{t_{\sigma k}-t_0} \\ +\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_{\sigma k}}\varphi(t_{\sigma(k+1)})\frac{t-t_0}{t_{\sigma(k+1)}-t_0} \\ -\frac{t-t_0}{t_{\sigma(k+1)}-t_{\sigma k}}(\frac{t_{\sigma(k+1)}-t}{t_{\sigma k}-t_0} \\ +\frac{t-t_{\sigma k}}{t_{\sigma(k+1)}-t_0})S_1(\varphi;t_0,\nu)\}\frac{1}{t-t_0}dt. \end{array} \right|$$

$$\simeq \left| \begin{array}{l} \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma k}t_{\sigma(k+1)}} \frac{\varphi(t_{\nu k})-\varphi(t_{\sigma k})}{t_{\nu k}-t_{\sigma k}} \\ +\frac{\varphi(t_{\nu(k+1)})-\varphi(t_{\sigma(k+1)})}{t_{\nu(k+1)}-t_{\sigma(k+1)}}dt \end{array} \right| = O(\frac{\ln MN}{M^\mu N^\mu}).$$

Naturally, the estimation given above is obtained with the help of the density φ taken as an element of the Hölder space $H(\mu)$ [2]. Further, for the cases where $\sigma = \nu-1, \nu+1$ and ν, we obtain using again the condition $\varphi \in H(\mu)$ and the condition of smoothness of Γ. With this, we obtain the following estimation:

$$\begin{aligned} &\mid \frac{1}{\pi i}\int_{t_\nu t_{\nu+1}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0}dt \mid \\ &\leq A\sum_{\kappa=0}^{M-1}\int_{s_{\nu\kappa}}^{s_{\nu(\kappa+1)}} \mid s-s_0\mid^{\mu-1}ds = O(N^{-\mu}) \end{aligned}$$

Numerical experiments

Using our approximation, we apply the algorithms to singular integrals, and we present results concerning the accuracy of the calculations. In this numerical experiments, Table 1 represents the exact principal value of the singular integral, and Table 2 corresponds to the approximate calculation produced by our approximation at point value interpolation.

Example 1. *Consider the singular integral,*

$$I = F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt,$$

where the curve Γ designates the unit circle, and the function density φ is given by the following expression:

$$\varphi(t) = \frac{1}{t-2}.$$

Example 2. *Consider the singular integral,*

$$I = F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt,$$

where the curve Γ designates the unit circle, and the function density φ is given by the following expression:

$$\varphi(t) = \frac{t}{t^2-4}$$

Conclusion

We remark for this approximation based on the trapezoidal technical convergence to the exact value of the singular integrals given by the Cauchy principal value. Also, these numerical calculations show that the accuracy improves with increased number of subdivisions.

Competing interests

The author declares that he has no competing interests.

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MN is a PhD degree holder and a full professor in the University of Msila.

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